The central limit theorem for stationary Markov processes with normal generator – with applications to hypergroups

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Abstract

We extend the central limit theorem for additive functionals of a stationary, ergodic Markov chain with normal transition operator due to Gordin and Lifšic [12] to continuous-time Markov processes with normal generators. As examples we discuss random walks on compact commutative hypergroups as well as certain random walks on non-commutative, compact groups.

Keywords: Central limit theorem; Hypergroup; Markov processes; Martingale; Normal generator; Random walk

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1 Introduction

The central limit theorem (CLT) for additive functionals of stationary, ergodic Markov chains has been studied intensively during the last decades. Let $(X_n)_{n\geq 0}$ be a stationary, ergodic Markov chain with state space $(X, \mathcal{B})$, transition operator Q and invariant initial distribution μ. A situation which is particularly well understood is that in which Q is a normal operator on $L_2^{\mathbb{C}}(\mu)$. This was first considered by Gordin and Lifšic [12]. Denote by L_2^0 the set of real-valued functions with $\int f\, d\mu = 0$ and let

$$S_n(f) = f(X_1) + \ldots + f(X_n)$$

be the partial sums. Assume that Q is normal and given $f \in L_2^0$ let ρ_f denote the spectral measure of Q with respect to f (cf. Ref. [2] for the definition). In [12] it is shown that if $f \in L_2^0$ satisfies

$$\int_{\sigma(Q)} \frac{1}{|1-z|}\, d\rho_f(z) < \infty, \tag{1}$$

then $S_n(f)/\sqrt{n}$ is asymptotically normal with variance

$$\sigma^2(f) = \int_{\sigma(Q)} \frac{1-|z|^2}{|1-z|^2}\, d\rho_f(z). \tag{2}$$

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It seems that at that time their result did not receive much attention, and complete proofs were only published later in [4]. Kipnis and Varadhan [16] reproved the result for reversible chains, which correspond to self-adjoint Q, using a different technique, and Deriennic and Lin [6] gave a proof for the normal case without use of the spectral theorem. They used the condition $f \in \mathrm{Im}(\sqrt{I-Q})$, which is equivalent to (1) (cf. Ref. [8]). In this paper we mainly consider continuous-time Markov processes. Let $(X_t)_{t\geq 0}$ be a stationary ergodic Markov process, defined on a probability space $(\Omega, \mathcal{A}, P)$, with state space $(X, \mathcal{B})$, transition probability function $p(t,x,dy)$ and stationary distribution μ. We assume that the contraction semigroup

$$T_t f(x) = \int_X f(y)\, p(t,x,dy), \quad f \in L_2(\mu),$$

is strongly continuous (on $L_2(\mu)$). Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration in $(\Omega, \mathcal{A}, P)$ such that $(X_t)_{t\geq 0}$ is progressively measurable with respect to $(\mathcal{F}_t)_{t\geq 0}$ and satisfies the Markov property

$$E\big(f(X_t)|\mathcal{F}_u\big) = T_{t-u} f(X_u), \quad f \in L_2(\mu), \quad 0 \leq u < t.$$

Let L be the generator of $(T_t)_{t\geq 0}$ and $\mathcal{D}(L)$ its domain of definition on $L_2^{\mathbb{C}}(\mu)$. Given $f \in L_2^0,\ t > 0$ let

$$S_t(f) = \int_0^t f(X_s)\, ds.$$

Without further assumptions on the generator L, Bhattacharya [1] proved asymptotic normality for $S_t(f)/\sqrt{t}$ (in fact even the functional CLT) if $f \in \mathrm{Im}(L)$. For a reversible Markov process (which corresponds to self-adjoint L), in Ref. [16] the CLT under the assumption that $f \in \mathrm{Im}(\sqrt{-L})$ is proved.

In this paper we study the case in which L is a normal operator, i.e. $LL^* = L^*L$. Recall that the generator L is normal if and only if each operator $T_t,\ t > 0$, of the corresponding semigroup is normal (cf. Ref. [19], p. 360). In Section 2 the CLT for Markov processes with normal generator L under a spectral assumption similar to (1) is proved, following the method used in [16] for the self-adjoint case. We point out that the method of Gordin and Lifšic [12] for discrete-time chains seems not to be applicable in continuous time. An interesting situation in which the generator L turns out to be normal but not necessarily self-adjoint is that of a convolution semigroup on a compact, commutative hypergroup. In Section 3 we prove a CLT for the corresponding random walks. Random walks on non-commutative compact groups, where the corresponding convolution semigroup is contained in the center of measure algebra, are discussed in Section 4.

2 The central limit theorem

In this section we will prove the CLT for stationary, ergodic Markov processes with normal generator. Assume that L is a normal operator on $L_2^{\mathbb{C}}(\mu)$ with spectrum $\sigma(L)$ and for $f \in L_2^0$ denote by $\rho_f(dz)$ the spectral measure of L with respect to f. Recall that we have $\Re(z) \leq 0$ for each $z \in \sigma(L)$. Consider the condition

$$\int_{\sigma(L)} \frac{1}{|z|}\, \rho_f(dz) < \infty. \tag{3}$$

Given $\epsilon > 0$ let $g_\epsilon = (\epsilon I - L)^{-1} f$ be the image under the resolvent mapping. Recall that $g_\epsilon \in \mathcal{D}(L)$, the domain of definition of L, for any $\epsilon > 0$. The norm in $L_2^{\mathbb{C}}(\mu)$ is denoted by $\|\cdot\|$ and the scalar product by $< \cdot, \cdot >$.

Lemma 1. *Assume that L is normal and that $f \in L_2^0$ satisfies (3). Then*

$$\lim_{\epsilon \to 0} \epsilon < g_\epsilon, g_\epsilon >= 0 \tag{4}$$

and

$$\lim_{\delta, \epsilon \to 0} < g_\epsilon - g_\delta - T_t(g_\epsilon - g_\delta),\, g_\epsilon - g_\delta >= 0. \tag{5}$$

Proof. In order to show (4), from the spectral theorem it follows that

$$\epsilon < g_\epsilon, g_\epsilon >= \int_{\sigma(L)} \frac{\epsilon}{|\epsilon - z|^2} \rho_f(dz).$$

Since $\Re(z) \leq 0$ for $z \in \sigma(L)$ we estimate

$$\begin{aligned} |\epsilon - z|^2 &= \epsilon^2 + |z|^2 - 2\epsilon \cdot \Re(z) \\ &\geq \epsilon^2 + |z|^2 \geq 2\epsilon |z|. \end{aligned}$$

Thus (4) follows from (3) and the dominated convergence theorem. As for (5), we have from the spectral theorem

$$\begin{aligned} < g_\epsilon - g_\delta - T_t(g_\epsilon - g_\delta),\, g_\epsilon - g_\delta > &= \int_{\sigma(L)} (1 - e^{zt}) \Big[\frac{1}{\epsilon - z} - \frac{1}{\delta - z}\Big] \Big[\frac{1}{\epsilon - \bar{z}} - \frac{1}{\delta - \bar{z}}\Big] \rho_f(dz) \\ &\leq \int_{\sigma(L)} |1 - e^{zt}| \frac{(\epsilon - \delta)^2}{|\epsilon - z|^2 \cdot |\delta - z|^2} \rho_f(dz). \end{aligned}$$

We can assume $\epsilon > \delta > 0$. Now $|\epsilon - z|^2 |\delta - z|^2 \geq |z|^2 \epsilon^2$. On $\sigma(L) \cap \{|z| \leq 1\}$ we have $|1 - e^{zt}| \leq |zt| e^t$, and the integrand is dominated by $te^t/|z|$. On $\sigma(L) \cap \{|z| > 1\}$ we have

$$|1 - e^{zt}| \leq 1 + |e^{zt}| = 1 + e^{\Re z\, t} \leq 2,$$

and the integrand is dominated by $2/|z|^2 \leq 2/|z|$. Again (5) follows from (3) and the dominated convergence theorem. □

Theorem 1. *Let $(X_t)_{t \geq 0}$ be a progressively measurable stationary ergodic Markov process with state space $(X, \mathcal{B})$, strongly continuous contraction semigroup $(T_t)_{t>0}$ and stationary distribution μ. Assume that the generator L is normal on $L_2^{\mathbb{C}}(\mu)$, and that $f \in L_2^0$ satisfies (3). Then*

$$\frac{S_t(f)}{\sqrt{t}} \overset{t \to \infty}{\Rightarrow} N(0, \sigma^2(f)),$$

where the limit variance satisfies

$$\sigma^2(f) = \lim_{t \to \infty} E\big(S_t(f)\big)^2 / t = -2 \int_{\sigma(L)} \frac{1}{z} \rho_f(dz).$$

Here $N(0, \sigma^2)$ denotes the normal law with mean 0 and variance σ^2, and $\Rightarrow$ denotes weak convergence of distributions.

Proof. Consider the decomposition

$$S_t(f) = M_{t,\epsilon} + \epsilon S_t(g_\epsilon) + A_{t,\epsilon},$$

where

$$\begin{aligned} M_{t,\epsilon} &= g_\epsilon(X_t) - g_\epsilon(X_0) - \int_0^t (Lg_\epsilon)(X_s)\, ds, \\ A_{t,\epsilon} &= -g_\epsilon(X_t) + g_\epsilon(X_0), \end{aligned}$$

and $(M_{t,\epsilon})_{t\geq 0}$ is a martingale with stationary increments and $M_{0,\epsilon} = 0$ for any $\epsilon > 0$. For any $h \in L_2(\mu)$, from the Schwarz inequality,

$$E\Big(\int_0^t h(X_s)\, ds\Big)^2 \leq E\Big(t \int_0^t h(X_s)^2\, ds\Big) = t^2 \|h\|^2. \tag{6}$$

From (6) and (4) it follows that $\epsilon^2 E S_t(g_\epsilon)^2 \to 0$. Furthermore, since

$$E(A_{t,\epsilon} - A_{t,\delta})^2 = 2 < g_\epsilon - g_\delta - T_t(g_\epsilon - g_\delta),\, g_\epsilon - g_\delta >,$$

the convergence of $A_{t,\epsilon}$ to some A_t as $\epsilon \to 0$ follows directly from (5) via the Cauchy criterion. Since $M_{t,\epsilon} = \epsilon S_t(g_\epsilon) + A_{t,\epsilon} - S_t(f)$, $M_{t,\epsilon}$ also converges to a limit M_t, which is also a martingale with stationary increments, and $S_t(f) = M_t + A_t$. Using (4) it is easy to show (see Kipnis and Varadhan [16]) that $EA_t^2/t \to 0$ as $t \to \infty$. Asymptotic normality follows from the CLT for martingales with stationary increments. This result is well-known for discrete-time martingales; see Chikin [5] for a careful discussion of the continuous-time case. Finally let us prove the formula for $\sigma^2(f)$. From [14],

$$EM_1^2 = \sigma^2(f) = \lim_{n\to\infty} 2n < g_{1/n} - T_{1/n} g_{1/n}, g_{1/n} > .$$

Now

$$2n < g_{1/n} - T_{1/n} g_{1/n}, g_{1/n} >= 2 \int_{\sigma(L)} \frac{1 - e^{z/n}}{1/n} \frac{1}{|1/n - z|^2}\, d\rho_f(z).$$

The integrand converges to $-1/\bar{z}$, and by an application of the dominated convergence theorem which can be justified as above the formula for $\sigma^2(f)$ follows. This finishes the proof of the theorem. $\square$

Remark 1. A functional CLT for Markov chains with normal transition operator, started at a point, was proved in Ref. [7] under a spectral assumption slightly stronger than (3). It would be of some interest to obtain a similar result for continuous-time Markov processes.

Remark 2. Kipnis and Varadhan [16] in fact obtained the functional central limit theorem for reversible Markov processes under the condition (3). A simpler proof of the functional part was given by Olla [18]. It is possible to deduce from his results that if

$$-\int_{\sigma(L)} \frac{1}{\Re z}\, d\rho_f(z) < \infty,$$

then the functional CLT holds in case of a stationary Markov process with normal generator. Whether such a result is already true under the milder spectral assumption (3) remains an open problem.

3 Random walks on compact commutative hypergroups

In this section we apply Theorem 1 to random walks on compact commutative hypergroups. Roughly speaking, a hypergroup is a Hausdorff space H such that the space of regular finite Borel measures $\mathcal{M}_b(H)$ can be equipped with a convolution operation which preserves the probability measures. Axiomatic schemes for this concept were first introduced by Dunkl [9] and Jewett [15]. Since then hypergroups have been investigated intensively, due to the rich variety of examples, and a rather general notion of hypergroups has become standard in the literature. Let H be a locally compact Hausdorff space. We denote by $\mathcal{M}_b(H)$ the space of regular finite Borel measures and by $\mathcal{M}_1(H)$ the subset of regular probabilities. Our definition of a hypergroup is taken from Bloom and Heyer [3].

Definition 1. H is called a *hypergroup* if the space $(\mathcal{M}_b(H), +)$ admits a second binary operation $*$ such that the following conditions are satisfied.

1. $(\mathcal{M}_b(H), +, *)$ is an algebra.
2. For any $x, y \in H$, $\delta_x * \delta_y \in \mathcal{M}_1(H)$ and $\mathrm{supp}(\delta_x * \delta_y)$ is compact (here δ_x denotes the Dirac measure at $x \in H$).
3. The mappings $(x, y) \mapsto \delta_x * \delta_y$ and $(x, y) \mapsto \mathrm{supp}(\delta_x * \delta_y)$ of $H \times H$ are continuous with respect to the weak topology and the Michael topology, respectively.
4. There exists an involution $x \mapsto \bar{x}$ of H such that $\overline{\delta_x * \delta_y} = \delta_{\bar{y}} * \delta_{\bar{x}}$ for all $x, y \in H$, where $\bar{\nu}$ denotes the image of $\nu \in \mathcal{M}_b(H)$ under the involution $\bar{}$.
5. There exists an element $e \in H$ such that $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$ for all $x \in H$, and such that $e \in\mathrm{supp}(\delta_x * \delta_y)$ if and only if $y = \bar{x}, \quad x, y, \in H$.

The hypergroup H is called *commutative* if $(\mathcal{M}_b(H), +, *)$ is a commutative algebra. In the following let H be a commutative hypergroup. The *x-translate* of a function $f \in C_c(H)$ is defined by

$$\tau_x f(y) = f(x * y) = \int_H f d(\delta_x * \delta_y).$$

A measure $\nu \in \mathcal{M}_b(H)$ is called *invariant* if

$$\int_H \tau_x f \, d\mu = \int_H f \, d\mu, \quad f \in C_c(H), \ x \in H.$$

A *compact* hypergroup (i.e. H is a compact) always admits a unique invariant measure $\mu \in \mathcal{M}_1(H)$ (cf. [3], p. 40), and we have the formula (cf. [3], p. 34)

$$\int_H f(x * y) g(y) d\mu(y) = \int_H f(y) g(\bar{x} * y) d\mu(y) \quad \forall \ f, g \in L_2^{\mathbb{C}}(\mu). \tag{7}$$

Furthermore, translation can be extended to the space $L_2^{\mathbb{C}}(\mu)$. The *convolution* of a function $f \in L_2^{\mathbb{C}}(\mu)$ and a measure $\nu \in \mathcal{M}_b(H)$ is defined by

$$f * \nu(x) = \int_K f(x * \bar{y}) \, d\nu(y).$$

A non-zero, continuous function $\chi : H \to \mathbb{C}$ is called a *character* if

$$\chi(x * \bar{y}) = \chi(x)\overline{\chi(y)}, \quad x, y \in H.$$

It follows that $\chi(e) = 1$, $|\chi(x)| \leq 1$ and $\chi(\bar{x}) = \overline{\chi(x)}$. The set of characters is denoted by $\hat{H}$. If H is compact and commutative, $\hat{H}$ is discrete (with respect to the topology of uniform convergence), and forms an orthogonal basis of $L_2^{\mathbb{C}}(\mu)$ (cf. [9], p. 340). The *Fourier transform* of a function $f \in L_2^{\mathbb{C}}(\mu)$ and of a measure $\nu \in \mathcal{M}_b(H)$ are defined respectively by

$$\hat{f}, \hat{\nu} : \hat{H} \to \mathbb{C}, \quad \hat{f}(\chi) = \int_H f \bar{\chi}\, d\mu, \qquad \hat{\nu}(\chi) = \int_H \bar{\chi}\, d\nu.,$$

The *Plancherel measure* on $\hat{H}$ is given by $\pi = \sum_{\chi \in H} c(\chi)\, \delta_\chi$, where δ_χ is the Dirac measure at x and

$$c(\chi) = \Big(\int_H |\chi|^2 d\mu \Big)^{-1}.$$

Furthermore we have the Plancherel formula and the inversion formula (cf. [3], pp. 86, 91).

Firstly let us consider discrete-time random walks. Let $Q \in \mathcal{M}_1(H)$ be a probability measure on H. Then we can define a Markov kernel Q on $L_2^{\mathbb{C}}(\mu)$ by letting $Qf(x) = f * Q(x)$. Using the translation invariance of the Haar measure one shows that this Markov kernel preserves μ. Now we are in the position to state the following result.

Theorem 2. *Let H be a compact, commutative hypergroup with Haar measure μ. Let $Q \in \mathcal{M}_1(H)$ and let $(X_n)_{n\geq 0}$ be a random walk in H with transition operator Q and stationary distribution μ. Suppose that 1 is a simple eigenvalue of Q and that $f \in L_2^0$ satisfies*

$$\sum_{\chi \in \hat{H}} \frac{1}{|1 - \hat{Q}(\chi)|} c(\chi) |\hat{f}(\chi)|^2 < \infty.$$

Then $S_n(f)/\sqrt{n}$ is asymptotically normally distributed, and the limit variance is given by

$$\sigma^2(f) = \sum_{\chi \in \hat{H}} \frac{1 - |\hat{Q}(\chi)|^2}{|1 - \hat{Q}(\chi)|^2}\, c(\chi) |\hat{f}(\chi)|^2.$$

Proof. We want to apply condition (1), as obtained by Gordin and Lifšic [12]. It is well-known that the chain $(X_n)_{n\geq 0}$ is ergodic if and only if 1 is a simple eigenvalue of Q. Now let us show that Q is a normal operator. To this end, using (7) the following is easily shown.

$$\int_H (Qf)(x)\overline{g(x)} d\mu(x) = \int_H f(x) \int_H \overline{g(x * y)}\, dQ(y)\, d\mu(x), \quad f, g \in L_2^{\mathbb{C}}(\mu).$$

Therefore the adjoint operator is given by $(Q^* g)(x) = \int g(x * y) dQ(y)$, i.e. by convolution with respect to the measure $\bar{Q}$. By commutativity it follows that Q is normal. Furthermore we have that

$$\chi * Q = \hat{Q}(\chi)\chi, \quad \chi \in \hat{H}. \tag{8}$$

Therefore Q has a discrete spectrum and each χ is an eigenvector with eigenvalue $\hat{Q}(\chi)$. The theorem now follows from (1) and (2). □

Remark 3. Related results on the central limit theorem for random walks on hypergroups, where H is a non-compact interval or the lattice $\mathbb{Z}$ or $\mathbb{Z}_+$, can be found in [11].

Now let us consider continuous-time random walks. A *convolution semigroup* $(Q_t)_{t>0} \subset \mathcal{M}_1(H)$ is a family of probability measures such that $Q_t * Q_s = Q_{s+t}$. It is called *e-continuous* (or simply *continuous*) if $\lim_{t\to 0} Q_t = \delta_e$ in the topology of weak convergence. For every e-continuous convolution semigroup there exists a negative definite function $\psi \in N_B^{(s)}(\hat{H})$ (see [3], p. 334), called the *exponent* of the convolution semigroup, such that $\hat{Q}_t = \exp(-t\psi)$. Given an e-continuous convolution semigroup, we obtain a contraction semigroup by letting $T_t = f * Q_t,\ f \in L_2^{\mathbb{C}}(\mu)$ (cf. [3], p. 427). This semigroup commutes with translations, and gives rise to a stationary Markov process $(X_t)_{t\geq 0}$ with stationary distribution μ. We have the following

Theorem 3. *Let H be a compact, commutative hypergroup with Haar measure μ. Let $(Q_t)_{t>0}$ be an e-continuous convolution semigroup with exponent $\psi \in N_B^{(s)}(\hat{H})$ and let $(X_t)_{t\geq 0}$ be the corresponding continuous-time random walk with semigroup T_t, generator L, and stationary distribution μ. Suppose that 0 is a simple eigenvalue of L and that $f \in L_2^0$ satisfies*

$$\sum_{\chi\in\hat{H}} \frac{1}{|\psi(\chi)|}\, c(\chi)|\hat{f}(\chi)|^2 < \infty. \tag{9}$$

Then $S_t(f)/\sqrt{t}$ is asymptotically normally distributed with limit variance

$$\sigma^2(f) = 2\sum_{\chi\in\hat{H}} \frac{1}{\psi(\chi)}\, c(\chi)|\hat{f}(\chi)|^2.$$

Proof. First let us show that the semigroup (T_t) is strongly continuous. In fact, the Fourier transform gives rise to the contraction semigroup on $L_2^{\mathbb{C}}(\hat{H},\pi)$ given by the multiplication operators $M_t F = \exp(-t\psi)F,\ F \in L_2^{\mathbb{C}}(\hat{H},\pi)$. Such contraction semigroups are always strongly continuous (cf. Nagel and Schlotterbeck [17], p. 8), and their generator is the densely-defined multiplication operator $\hat{L}F = -\psi F$. Thus from the Fourier isometry, it follows that the generator L of (T_t) is also densely defined with domain

$$\mathcal{D}(L) = \{f \in L_2^{\mathbb{C}}(H,\mu): \quad \psi\hat{f} \in L_2^{\mathbb{C}}(\hat{H},\pi)\},$$

and

$$\widehat{(Lf)} = -\psi\hat{f}, \quad f \in \mathcal{D}(L).$$

For $f = \chi$ with $\chi \in \hat{H}$ this gives

$$\widehat{(L\chi)}(\gamma) = -\psi(\chi)c(\chi)^{-1}\mathbb{1}_{\{\chi\}}(\gamma), \quad \chi,\gamma \in \hat{H}.$$

From the inversion theorem ([3], pp. 89 - 92) we get that

$$L\chi = -\psi(\chi)\chi.$$

The theorem follows from Theorem 1. □

Remark 4. Observe that L is self-adjoint if and only if $Q_t = \bar{Q}_t$ for all $t > 0$.

Example 1 (*Compact Abelian groups)***.** In this examples we illustrate the use of Theorems 2 and 3 by considering random walks on a separable compact Abelian group G. Let Γ denote the dual group of G and let μ_G be the normalized Haar measure. It is well known that characters form an orthonormal basis of $L_2^{\mathbb{C}}(G)$. There is a hypergroup structure on G given by the usual convolution, i.e. $\delta_x * \delta_y = \delta_{x+y}$. Thus Haar measure on the hypergroup is the usual Haar measure on G, and the characters of the hypergroup are given by the characters of the group. Theorems 2 and 3 apply, and $c(\chi) = 1$ for all $\chi \in \Gamma$. In discrete time, this example was studied by Gordin & Lifšic ([4], pp. 171,72). Given an e-continuous convolution semigroup, the generating functional ψ can be decomposed as follows:

$$\psi = \psi_1 + \psi_2 + \psi_3,$$

where ψ_1 is a continuous primitive form, ψ_2 a continuous square form, and ψ_3 is given in terms of the Lévy function and the Lévy measure (see Heyer [13], pp. 70, 308). Let us consider the one-dimensional torus $\mathcal{T}^1$, where characters are of the form $\chi_n(\theta) = e^{in\theta},\ \theta \in [0, 2\pi)$. In this case (cf. Zimple [20], p. 493),

$$\psi_1(\chi_n) = -ian, \quad \psi_2(\chi_n) = bn^2, \quad a \in \mathbb{R},\ b \geq 0.$$

If $\psi = \psi_1$, $X_t = e^{iat}$ is a deterministic motion. As can be expected, (9) is satisfied for any $f \in L_2^0$ but $\sigma^2(f) = 0$. If $\psi = \psi_2$, the Q_t are wrapped Gaussian distributions with densities

$$q_t(\theta) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{-tn^2 b} \cos(n\theta).$$

(9) is also satisfied for any $f \in L_2^0$, and $\sigma^2(f) \neq 0$ if $f \neq 0$ (and $b \neq 0$). Notice that L is self-adjoint in this case. If the Lévy measure α is bounded, then

$$\psi_3(\chi_n) = \int_{G \setminus \{e\}} \big(1 - \chi_n(\theta)\big)\, d\alpha(\theta).$$

In this case (as well as in the case of general ψ), asymptotic normality depends on the Fourier expansion of $f \in L_2^0$.

4 Random walks on compact, non-Abelian groups

In this section we show how to apply Theorem 1 to certain random walks on compact, possibly non-Abelian groups.

Let G be a compact, separable group with normalized Haar measure μ_G and let $\hat{G}$ denote the set of equivalence classes of irreducible unitary representations of G. If $\alpha \in \hat{G}$, we let α also stand for some representative of this equivalence class, acting on a space V_α of finite dimension n_α. We have the orthogonal Hilbert space decomposition

$$L_2^{\mathbb{C}}(G) = \oplus_{\alpha \in \hat{G}} H_\alpha, \quad H_\alpha = \{g \mapsto tr(\alpha(g)C),\ g \in G,\ C \in \mathrm{End}(V_\alpha)\}, \quad \alpha \in \hat{G},$$

where $tr(C)$ denotes the trace of the endomorphism C (cf. Ref. [10]). The orthogonal projection of $f \in L_2^{\mathbb{C}}(G)$ to H_α is given by $n_\alpha f_\alpha$, where $f_\alpha = f * \chi_\alpha = \chi_\alpha * f$, and χ_α is the character of α.

Let H be the set of conjugacy classes with the quotient topology. There is a one-to-one correspondence between $\mathcal{M}_b(H)$ and $\mathcal{Z}(\mathcal{M}_b(G))$, the center of $\mathcal{M}_b(G)$. Therefore H can be equipped with a commutative hypergroup structure, and Theorems 2 and 3 apply to random walks on H. Explicitly, the characters of H are given by the normalized characters of the group $\gamma_\pi = \chi_\pi / n_\alpha$.

We want to extend this result to functions which are not necessarily conjugation-invariant. Notice that $Q \in \mathcal{Z}(\mathcal{M}_b(G))$ is ergodic on $L_2^{\mathbb{C}}(G)$ if and only if it is ergodic on $L_2^{\mathbb{C}}(H, \mu)$, since 0 is either a simple or multiple eigenvalue in both cases.

Theorem 4. *Let G be a compact, separable, non-Abelian group and let Q be a probability on G. Suppose that $Q \in \mathcal{Z}(\mathcal{M}^b(G))$ and that Q, as a convolution operator, has 1 as a simple eigenvalue. Let $(X_n)_{n\geq 0}$ be a random walk on G with transition operator Q and stationary distribution μ_G. If $f \in L_2^0$ satisfies*

$$\sum_{\alpha \in \hat{G}} \frac{1}{|1 - \hat{Q}(\chi_\alpha)/n_\alpha|} n_\alpha^2 \|f_\alpha\|^2 < \infty,$$

then there is a martingale approximation to $S_n(f)$, where the limit variance is given by

$$\sum_{\alpha \in \hat{G}} \frac{1 - |\hat{Q}(\chi_\alpha)/n_\alpha|^2}{|1 - \hat{Q}(\chi_\alpha)/n_\alpha|^2} n_\alpha^2 \|f_\alpha\|^2 < \infty.$$

Proof. Since $Q \in \mathcal{Z}(\mathcal{M}_b(G))$, from (8) we obtain $Q * \gamma_\alpha = \hat{Q}(\gamma_\alpha)\gamma_\alpha$ or $Q * \chi_\alpha = \hat{Q}(\chi_\alpha)/n_\alpha \, \chi_\alpha$. Given any $f \in L_2^{\mathbb{C}}(G)$ and $\alpha \in \hat{G}$, we have since $Q \in \mathcal{Z}(\mathcal{M}^b(G))$,

$$Q * f_\alpha = Q * f * \chi_\alpha = f * Q * \chi_\alpha = \hat{Q}(\chi_\alpha)/n_\alpha \, f * \chi_\alpha = \hat{Q}(\chi_\alpha)/n_\alpha \, f_\alpha.$$

Therefore, each space H_α is an eigenspace of Q with eigenvalue $\hat{Q}(\chi_\alpha)/n_\alpha$ and in particular, Q is a normal operator. The theorem follows from condition (1), due to Gordin and Lifšic [12]. □

A similar result can be formulated for e-continuous convolution semigroups in $\mathcal{Z}(\mathcal{M}^b(G))$.

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